MATH 2443
Final Exam Review Sheet Solutions

1. Assume that the function $f(u, v)$ has continuous partial derivatives f_u and f_v and suppose that $f_u(1, 1) = 1$ and that $f_v(1, 1) = 2$. A new function $g(x, y, z)$ is defined by setting $g(x, y, z) = f(x/y, y/z)$. Compute $g_y(1, 1, 1)$.

   Set $u = x/y$ and $v = y/z$. By the chain rule,
   $g_y(x, y, z) = f_u(u, v)\partial u/\partial y + f_v(u, v)\partial v/\partial y = f_u(u, v)(-x/y^2) + f_v(u, v)(1/z)$.
   When $x = y = z = 1$, we have that $u = v = 1$ so
   $g_y(1, 1, 1) = f_u(1, 1)(-1) + f_v(1, 1) = 1(-1) + 2 = 1$.

2. Evaluate the limit or show it does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4 + y^4}$.

   Along the path $x = 0$, this is $\displaystyle\lim_{y\to 0} \frac{0}{y^4} = 0$. Along the path $y = x$, this is $\displaystyle\lim_{x\to 0} \frac{6x^4}{3x^4} = 2$. As $0 \neq 2$, the limit does not exist.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{x^2 + 2y^2}$.

   Switching to polar we get that this equals

   $$\lim_{r\to 0} \frac{r^2 \cos^2(\theta) \sin^2(r \sin(\theta))}{r^2 \cos^2(\theta) + 2r^2 \sin^2(\theta)} = \lim_{r\to 0} \frac{\cos^2(\theta) \sin^2(r \sin(\theta))}{\cos^2(\theta) + 2 \sin^2(\theta)} .$$

   The bottom of this fraction is never 0 and the top approaches 0 as $r \to 0$ so the limit is 0.

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$.

   Switching to polar, we get $\displaystyle\lim_{r\to 0} \frac{\sin(r^2)}{r^2}$. By L'Hopitals rule, this is $\displaystyle\lim_{r\to 0} \frac{2r \cos(r^2)}{2r} = \lim_{r\to 0} \cos(r^2) = 1$.

   (d) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{y - 2x}{4 - xy^2}$.

   Along the path $x = 1$, this is $\displaystyle\lim_{y\to 2} \frac{y - 2}{4 - y^2} = \lim_{y\to 2} -\frac{1}{2 + y} = -\frac{1}{4}$. Along the path $y = 2$, this is $\displaystyle\lim_{x\to 1} \frac{2 - 2x}{4 - 4x} = \frac{1}{2}$. So the limit does not exist.

3. Find all critical points of the function f and determine if each critical point is a local max, local min, or saddle point.

(a) $f(x, y) = x^3y + 12x^2 - 8y$

The partial derivatives are $f_x(x, y) = 3x^2y + 24x$ and $f_y(x, y) = x^3 - 8$. The critical points are where both partial derivatives are 0 so $x^3 - 8 = 0$ which implies that $x = 2$ and $3x^2y + 24x = 0$ so $12y + 48 = 0$ and $y = -4$. So f has one critical point at $(2, -4)$.

To determine what type of point it is, use the second derivative test. The second order partial derivatives are $f_{xx}(x, y) = 6xy + 24, f_{yy}(x, y) = 0, f_{xy}(x, y) = 3x^2$ which are -24, 0, and 12 respectively when $x = 2, y = -4$. Then $D = f_{xx}f_{yy} - (f_{xy})^2 = -144$ so it is a saddle point.

(b) $f(x, y) = e^{4y-x^2-y^2}$

The partial derivatives are $f_x(x, y) = -2xe^{4y-x^2-y^2}, f_y(x, y) = (4 - 2y)e^{4y-x^2-y^2}$. As $e^{4y-x^2-y^2}$ is never 0, these are both 0 when $x = 0$ and $y = 2$ so f has one critical point at $(0, 2)$.

The second order partial derivatives are $f_{xx}(x, y) = e^{4y-x^2-y^2}(4x^2 - 2), f_{yy}(x, y) = e^{4y-x^2-y^2}((4 - 2y)^2 - 2)$, and $f_{xy}(x, y) = e^{4y-x^2-y^2}(-2x)(4 - 2y)$. These are $-2e^4, -2e^4$, and 0 respectively at $x = 0, y = 2$ so $D = 4e^8$. As $D > 0$ and $f_{xx} < 0$ this is a local maximum.

4. Find the point or points on the curve $x^2 + 3y^2 = 36$ which are closest to the point $(2, 0)$. Find the point or points on the curve which are furthest from the point $(2, 0)$.

We are trying to find max/mins of $\sqrt{(x-2)^2 + y^2}$ under the constraint $x^2 + 3y^2 = 36$. The maximum and minimum values of $\sqrt{(x-2)^2 + y^2}$ occur at the same place as the maximum and minimums of $(x - 2)^2 + y^2$ so instead we will look for max/mins of $f(x, y) = (x - 2)^2 + y^2$ under the constraint $g(x, y) = x^2 + 3y^2 = 36$. The region $x^2 + 3y^2 = 36$ is an ellipse which is closed and bounded so we are guaranteed to have an absolute maximum and minimum. We can find the max and min by finding all critical points and comparing the values of f at these points.

We use Lagrange multipliers to find the critical points. We need to find solutions to the three equations $f_x = \lambda g_x, f_y = \lambda g_y, g(x, y) = 36$ which are $2(x - 2) = \lambda 2x$, $2y = \lambda 6y$, and $x^2 + 3y^2 = 36$. By the second equation, we have that $y = 0$ or $\lambda = 1/3$. If $y = 0$ then by the third equation $x^2 = 36$ so $x = \pm 6$ and we get the critical points $(\pm 6, 0)$. If $\lambda = 1/3$ then the first equation becomes $2(x - 2) = (1/3)(2x)$ which means that $x = 3$. Then by the last equation $3^2 + 3y^2 = 36$ so $y = \pm 3$ and we get critical points $(3, \pm 3)$.

Plugging each critical point into f gives
$f(6,0) = 16, f(-6,0) = 64, f(3,3) = 10, f(3,-3) = 10$ so the closest points are $(3,3)$ and $(3,-3)$ and the furthest point is $(-6,0)$.

5. Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{y^2}{(x^2+y^2)^{3/2}}\, dydx.$

The curve $y = \sqrt{2x - x^2}$ can be rewritten as $x^2 - 2x + y^2 = 0$ and completing the square makes it $(x-1)^2 + y^2 = 1$ so it is the circle of radius 1 with center $(1,0)$. The region we are integrating over is the upper half of this circle. This integral will be much easier in polar. The circle $x^2 + y^2 = 2x$ becomes
$r^2 = 2r\cos(\theta)$ or $r = 2\cos(\theta)$. The θ values which trace out the upper half of the circle are from 0 to $\pi/2$ so the integral becomes

$$\int_0^{\pi/2} \int_0^{2\cos(\theta)} \frac{r^2 \sin^2(\theta)}{r^3} r\ drd\theta = \int_0^{\pi/2} \int_0^{2\cos(\theta)} \sin^2(\theta)\ drd\theta$$

$$= \int_0^{\pi/2} 2\cos(\theta)\sin^2(\theta)\ d\theta = \frac{2}{3}\sin^3(\theta)\Big|_0^{\pi/2} = \frac{2}{3}\ .$$

6. Let $f(x,y,z)$ be differentiable. Suppose $f(1,3,5) = 7$ and
$\nabla f(1,3,5) = \langle 2,-3,1\rangle$.

   (a) Compute the directional derivative of f at the point $(1,3,5)$ in the direction of the point $(-1,4,7)$.

   The direction vector is $\langle -2,1,2\rangle$ and the unit direction vector is $\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\rangle$. The directional derivative is the dot product of the unit direction vector with the gradient so it is $\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\rangle \cdot \langle 2,-3,1\rangle = -\frac{5}{3}$.

   (b) Find the equation of the tangent plane to the surface $f(x,y,z) = 7$ at the point $(1,3,5)$.

   The is the plane through the point $(1,3,5)$ with normal
   $\nabla f(1,3,5) = \langle 2,-3,1\rangle$ so the equation is
   $2(x-1) - 3(y-3) + (z-5) = 0$ or $2x - 3y + z = -2$.

   (c) Use linear approximation to estimate $f(.9, 3.2, 5.1)$.

   $f(.9,3.2,5.1) \approx f(1,3,5) + f_x(1,3,5)(.9-1) + f_y(1,3,5)(3.2-3) + f_z(1,3,5)(5.1-5) = 7 + 2(-.1) + (-3)(.2) + 1(.1) = 6.3$.

   (d) Compute $\nabla g(3,2)$ where $g(x,y) = f(2y-x, xy-3, x+y)$.

   Since we are using x,y for the variables plugged into g, we will use u,v,w for the variables in f. Then $g = f(u,v,w)$ where
   $u = 2y - x, v = xy - 3, w = x + y$. We need to find $g_x(3,2)$ and $g_y(3,2)$. By the chain rule,

$$g_x(x,y) = f_u(u,v,w)\frac{\partial u}{\partial x} + f_v(u,v,w)\frac{\partial v}{\partial x} + f_w(u,v,w)\frac{\partial w}{\partial x}$$

$$= f_u(u,v,w)(-1) + f_v(u,v,w)(y) + f_w(u,v,w)(1)$$

and

$$g_y(x,y) = f_u(u,v,w)\frac{\partial u}{\partial y} + f_v(u,v,w)\frac{\partial v}{\partial y} + f_w(u,v,w)\frac{\partial w}{\partial y}$$
$$= f_u(u,v,w)(2) + f_v(u,v,w)(x) + f_w(u,v,w)(1) \ .$$

When $x = 3, y = 2$ we have that $u = 1, v = 3, w = 5$ so
$g_x(3,2) = 2(-1) + (-3)(2) + (1)(1) = -7$ and
$g_y(3,2) = (2)(2) + (-3)(3) + (1)(1) = -4$. So $\nabla g(3,2) = \langle -7, -4\rangle$.

7. Find the area of the part of the surface $z = x^2 + y^2$ between the planes $z = 1$ and $z = 2$.

   The surface can be parametrized as $r(x,y) = \langle x, y, x^2 + y^2\rangle$ where $1 \le x^2 + y^2 \le 2$. The partial derivatives are $r_x = \langle 1, 0, 2x\rangle$ and $r_y = \langle 0, 1, 2y\rangle$ and their cross product is $r_x \times r_y = \langle -2x, -2y, 1\rangle$. The surface area will be the integral of $|r_x \times r_y|$ over the possible x, y values and $|r_x \times r_y| = \sqrt{4x^2 + 4y^2 + 1}$ so the surface area is

$$\iint_{1\le x^2+y^2\le 2} \sqrt{4x^2 + 4y^2 + 1}\ dA \ .$$

   Switching to polar this becomes $\int_0^{2\pi}\int_1^{\sqrt{2}} r\sqrt{4r^2+1}\ drd\theta$. Using $u = 4r^2 + 1, du = 8rdr$ we get that the inside integral is $\int_1^{\sqrt{2}} r\sqrt{4r^2+1}\ dr = \int_5^9 \frac{1}{8}\sqrt{u}\ du = \frac{1}{12}u^{3/2}\Big|_5^9 = \frac{1}{12}(27 - 5\sqrt{5})$ so the double integral is $\int_0^{2\pi} \frac{1}{12}(27 - 5\sqrt{5})\ d\theta = \frac{(27-5\sqrt{5})\pi}{6}$.

8. Evaluate $\displaystyle\int_0^1\int_1^4 x\sqrt{3 + x^2/y}\ dydx + \int_1^2\int_{x^2}^4 x\sqrt{3 + x^2/y}\ dydx$.

   This would be difficult to integrate in this order, but could be integrated in the order $dxdy$ using u-substitution. These two regions can be combined into one $dxdy$ region as $\int_1^4\int_0^{\sqrt{y}} x\sqrt{3 + x^2/y}\ dxdy$. Using the u-substitution $u = 3 + x^2/y$, $du = (2x/y)dx$ the inside integral is $\int_0^{\sqrt{y}} x\sqrt{3 + x^2/y}\ dx = \int_3^4 (y/2)\sqrt{u}\ du = (y/3)u^{3/2}\Big|_3^4 = (\frac{8}{3} - \sqrt{3})y$. So the double integral is $\int_1^4 (\frac{8}{3} - \sqrt{3})y\ dy = (\frac{1}{2})(\frac{8}{3} - \sqrt{3})y^2\Big|_1^4 = (\frac{1}{2})(\frac{8}{3} - \sqrt{3})(16 - 1) = \frac{40-15\sqrt{3}}{2}$.

9. The function $f(x,y,z) = x^2 + 2xy + 2y^2 + 3z^2$ has a minimum value on the plane $x + 3y + 3z = 8$. Find the point where the minimum occurs.

   Using Lagrange multipliers with $g(x,y,z) = x + 3y + 3z$ we need to find solutions to the four equations $f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, x + 3y + 3z = 8$.

The equations are $2x + 2y = \lambda, 2x + 4y = 3\lambda, 6z = 3\lambda, x + 3y + 3z = 8$. Subtracting the first equation from the second we get that $2y = 2\lambda$ so $y = \lambda$. Plugging this into the first equation we get that $x = -\frac{1}{2}\lambda$. The third equations gives us that $z = \frac{1}{2}\lambda$. We plug all of these into the fourth equation to get $4\lambda = 8$ so $\lambda = 2$ then use this to get that $x = -1, y = 2, z = 1$. The only critical point is $(-1, 2, 1)$ so the minimum must occur at this point.

10. Let $w = x\sqrt{y} - x - y$. Find the maximum and minimum values of w and where the occur on the triangular region bounded by the x-axis, the y-axis, and the line $x + y = 12$.

    This is closed and bounded region so we will find all critical points and find w at each. First check for interior critical points where both partial derivatives are 0. The partial derivatives are $\partial w/\partial x = \sqrt{y} - 1$ and $\partial w/\partial y = \frac{x}{2\sqrt{y}} - 1$ which are both 0 at the point $(2, 1)$. This point is in the region so it is our only interior critical point.

    Next check for critical points on each of the three boundary lines. On $x = 0$, $w = -y$ and $w' = -1$ so there are no critical points. Similarly on $y = 0$, $w = -x$ so there are no critical points. For the line $x + y = 12$ we will use Lagrange multipliers. The equations we get are $\sqrt{y} - 1 = \lambda, \frac{x}{2\sqrt{y}} - 1 = \lambda, x + y = 12$. Combining the first two equations we get that $x = 2y$ and plugging this into $x + y = 12$ we get that $y = 4$ and $x = 8$. This point is in our region so we have a critical point at $(8, 4)$.

    Finally, we must also include corner points where two boundary curves meet. This gives us three more critical points: $(0, 0)$, $(0, 12)$, and $(12, 0)$. We check the value of w at all 5 critical points.

Point	w
$(2, 1)$	-1
$(8, 4)$	4
$(0, 0)$	0
$(12, 0)$	-12
$(0, 12)$	-12

    The maximum is 4 at $(8, 4)$ and the minimum is -12 at $(0, 12)$ and $(12, 0)$.

11. Given a function $f(x, y)$, suppose its gradient at the point $(1, 2)$ is $\langle 2, -4 \rangle$.

    (a) Find the directional derivative of f in the direction of the origin.

    The direction is $\langle -1, -2 \rangle$ and the unit vector in that direction is $\frac{1}{\sqrt{5}}\langle -1, -2 \rangle$. Dot with the gradient to get the directional derivative is $\frac{1}{\sqrt{5}}\langle -1, -2 \rangle \cdot \langle 2, -4 \rangle = \frac{6}{\sqrt{5}}$.

(b) Find the directional derivative of f in the direction of the maximum rate of increase of f.

The direction of the maximum rate of increase is the direction of the gradient and the directional derivative in that direction is $|\nabla f(1,2)| = \sqrt{20}$.

(c) Let $w = f(t^3, t^2+1)$. Find dw/dt at $t = 1$.

Write $x = t^3, y = t^2+1$. By the chain rule, $dw/dt = f_x(x,y)(dx/dt) + f_y(x,y)(dy/dt) = f_x(x,y)(3t^2) + f_y(x,y)(2t)$. When $t = 1$, $x = 1, y = 2$ so this is $f_x(1,2)(3) + f_y(1,2)(2) = (2)(3) + (-4)(2) = -2$.

12. Find $\int_C y^2(e^x+1)dx + 2y(e^x+1)dy$ where C is the closed path formed of three parts: the curve $y = x^2$ from $(0,0)$ to $(2,4)$, the line segment from $(2,4)$ to $(0,2)$ and the line segment from $(0,2)$ to $(0,0)$.

C is closed and oriented counterclockwise so we can use Green's Theorem to evaluate this with $P = y^2(e^x+1), Q = 2y(e^x+1)$. We integrate $Q_x - P_y = (2ye^x) - (2y(e^x+1)) = -2y$. So the integral becomes $\int_0^2 \int_{x^2}^{x+2} -2y \; dydx = \int_0^2 -y^2\Big|_{x^2}^{x+2} = \int_0^2 -(x+2)^2 + x^4 \; dx = -\frac{1}{3}(x+2)^3 + \frac{1}{5}x^5\Big|_0^2 = -\frac{184}{15}$.

13. A particle is moved in the plane from the origin to the point $(1,1)$. While it is moving, it is acted on by the force $F = \langle y^2 - ye^x + xy, 2xy - e^x + x^2\rangle$. This experiment is done twice. The first time the particle is moved in a straight line and the second time is it moved along the curve $y = x^3$. The work done by the force the first time is W_1 and the second time it is W_2. Determine which of W_1 and W_2 is bigger and by how much.

Write C_1 for the line segment from $(0,0)$ to $(1,1)$ which is along the line $y = x$ and C_2 for the curve $y = x^3$ from $(0,0)$ to $(1,1)$. Then $C = C_2 \cup -C_1$ is a closed curve oriented counterclockwise. If W is the work done over C then $W = W_2 - W_1$. We don't need to know W_1 and W_2, just their difference so this is exactly what we want to calculate. We can calculate W using Green's theorem with $P = y^2 - ye^x + xy$ and $Q = 2xy - e^x + x^2$. Then as $Q_x = 2y - e^x + 2x, P_y = 2y - e^x + x$ the difference $Q_x - P_y$ is x. Then $W = \int_C F \cdot \; dr = \int_0^1 \int_{x^3}^x x \; dydx = \int_0^1 x^2 - x^4 \; dx = \frac{2}{15}$. So W_2 is larger than W_1 by $2/15$.

14. The force $F = \langle e^{x^2}, 2x - e^{y^2}\rangle$ acts on a particle moving from $(0,0)$ to $(1,1)$.

(a) Compute the work done by the force if the particle moves in a straight line.

We can parametrize the line segment C_1 as $x = t, y = t, 0 \leq t \leq 1$. Then $dx = dt, dy = dt$ so the work is
$\int_{C_1} e^{x^2} dx + (2x - e^{y^2})dy = \int_0^1 e^{t^2} + 2t - e^{t^2}\ dt = \int_0^1 2t\ dt = 1$.

(b) Compute the work done if the particle moves first along the x-axis to $(4, 0)$ then then in a straight line to $(1, 1)$.

Call this path C_2 and let $C = C_2 \cup -C_1$ where C_1 is the path from part a. C is a closed curve oriented counterclockwise so we can use Green's Theorem to find the work on C. Write D for the triangular region enclosed by C. We take $P = e^{x^2}$ so $P_y = 0$ and $Q = 2x - e^{y^2}$ so $Q_x = 2$ and by Green's Theorem $\int_C F \cdot\ dr = \iint_D 2\ dA = 2A(D)$ where $A(D) = 2$ is the area of the triangle D so $\int_C F \cdot\ dr = 4$. We then have that
$\int_C F \cdot\ dr = \int_{C_2} F \cdot\ dr - \int_{C_1} F \cdot\ dr$ so
$\int_{C_2} F \cdot\ dr = \int_C F \cdot\ dr + \int_{C_1} F \cdot\ dr = 4 + 1 = 5$.

15. Let $w = f(x, y, z)$ be a differentiable function. At the point $x = 3, y = 2, z = 1$ assume that $w = 4, \partial w/\partial x = -1, \partial w/\partial y = 2$, and $\partial w/\partial z = 3$. Now view z as a function of x and y implicitly defined by $f(x, y, z) = 4$. Find ∇z at $x = 3, y = 2$.

As we have assumed w is fixed to be 4, we have that changing x does not change w so by the chain rule $0 = f_z(x, y, z)(\partial z/\partial x) + f_x(x, y, z)$. When $x = 3, y = 2$ we have that $z = 1$ so this becomes $0 = 3(\partial z/\partial x) + (-1)$ so $\partial z/\partial x = \frac{1}{3}$. Similarly, $0 = f_y(3, 2, 1) + f_z(3, 2, 1)(\partial z/\partial y)$ so $\partial z/\partial y = -\frac{2}{3}$ and $\nabla z(3, 2) = \langle \frac{1}{3}, -\frac{2}{3} \rangle$.

16. Find the maximum and minimum of $f(x, y, z) = xy + \frac{1}{3}z^3$ on $x^2 + y^2 + 2z^2 \leq 32$.

This is a closed and bounded region so we need to find all critical points and evaluate f at each one. First look for interior critical points where all three partial derivatives are 0. The partial derivatives are $f_x = y, f_y = x, f_z = z^2$ so there is a critical point at $(0, 0, 0)$. This is in our region so we have exactly one interior critical point.

Next look for critical points on the boundary curve $x^2 + y^2 + 2z^2 = 32$ using Lagrange multipliers. The Lagrange multiplier equations are
$y = 2\lambda x, x = 2\lambda y, z^2 = 4\lambda z, x^2 + y^2 + 2z^2 = 32$. If we plug the first equation into the second we get that $x = (2\lambda)^2 x$ so $x = 0$ or $\lambda = \pm\frac{1}{2}$. We consider each of these three cases.

Case 1: $x = 0$. If $x = 0$ then by the first equation we have that $y = 0$ and by the fourth equation $2z^2 = 32$ so $z = \pm 4$ and we get the critical points $(0, 0, 4)$ and $(0, 0, -4)$.

Case 2: $\lambda = \frac{1}{2}$. The first equation becomes $y = x$ and the third equation is $z^2 = 2z$. This means that $z = 0$ or $z = 2$ and this case splits into two more cases. If $z = 0$ then plugging in $z = 0, y = x$ into the fourth equation we get that $2x^2 = 32$ so $x = \pm 4$ and there are critical points at $(4, 4, 0)$ and $(-4, -4, 0)$. If $z = 2$ then plugging in $y = x$ and $z = 2$ into the fourth equation we get that $2x^2 + 8 = 32$ so $x = \pm\sqrt{12}$ and we get the critical points $(\sqrt{12}, \sqrt{12}, 2)$ and $(-\sqrt{12}, -\sqrt{12}, 2)$.

Case 3: $\lambda = -\frac{1}{2}$. The first equation becomes $y = -x$ and the third becomes $z^2 = -2z$ so $z = 0$ or $z = -2$. As in the previous cases we plug $y = -x$ and $z = 0$ or $z = 2$ into the fourth equation and get the critical points $(4, -4, 0), (-4, 4, 0), (\sqrt{12}, -\sqrt{12}, -2), (-\sqrt{12}, \sqrt{12}, -2)$.

There are a total of 11 critical points and we evaluate f at each one.

Point	$f(x, y, z) = xy + \frac{1}{3}z^3$
$(0, 0, 0)$	0
$(0, 0, 4)$	64/3
$(0, 0, -4)$	-64/3
$(4, 4, 0)$	16
$(-4, -4, 0)$	16
$(4, -4, 0)$	-16
$(-4, 4, 0)$	-16
$(\sqrt{12}, \sqrt{12}, 2)$	44/3
$(-\sqrt{12}, -\sqrt{12}, 2)$	44/3
$(\sqrt{12}, -\sqrt{12}, -2)$	-44/3
$(-\sqrt{12}, \sqrt{12}, -2)$	-44/3

The maximum is 64/3 and the minimum is $-64/3$.

17. Set up but do not evaluate an integral equal to the area of that part of the surface $z = \sqrt{1 - x - y}$ that lies inside the cylinder of radius 1 whose axis is the x-axis.

    The cylinder has formula $y^2 + z^2 = 1$ so the part of the surface which is inside the cylinder will be where $y^2 + z^2 \le 1$. This is a condition on y and z so it will be easiest if we solve the surface for x. This will also get rid of the square root, but we need to remember that we only have positive z values. The surface becomes $x = 1 - y - z^2$ with $z \ge 0, y^2 + z^2 \le 1$. This can be parametrized as $r(y, z) = \langle 1 - y - z^2, y, z\rangle$. The partial derivatives are $r_y = \langle -1, 1, 0\rangle$ and $r_z = \langle -2z, 0, 1\rangle$ and the cross product is $r_y \times r_z = \langle 1, 1, 2z\rangle$. The surface area is the integral of $|r_y \times r_z| = \sqrt{4z^2 + 2}$ over the possible y, z values. The y, z values are $z \ge 0$ and $y^2 + z^2 \le 1$ so the surface area is $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \sqrt{4z^2 + 2}\, dzdy$.

18. Evaluate $\int_C (y + \sin(x))dx + (z^2 + \cos(y))dy + x^3 dz$ where C is the curve parametrized by $r(t) = \langle \sin(t), \cos(t), \sin(2t)\rangle$, $0 \le t \le 2\pi$. Hint: C is on the surface $z = 2xy$.

Computing this directly would be difficult so we will use Stokes' Theorem. By the trig identity $\sin(2t) = 2\sin(t)\cos(t)$ we see that C is on the surface $z = 2xy$. It is also on the surface $x^2 + y^2 = 1$ so we can think of C as the intersection of the surface $z = 2xy$ and $x^2 + y^2 = 1$. For Stokes' Theorem we need a surface with boundary curve C so we will take S to be the part of the surface $z = 2xy$ which is inside the cylinder $x^2 + y^2 = 1$. Note that C is oriented clockwise when viewed from above so we will take S to have downward orientation. By Stokes' Theorem, this integral is equal to $\iint_S \operatorname{curl}(F) \cdot\ d\mathbf{S}$.

S can be parametrized as $r(x, y) = \langle x, y, 2xy\rangle$ for $x^2 + y^2 \leq 1$. Then $r_x = \langle 1, 0, 2y\rangle, r_y = \langle 0, 1, 2x\rangle$ and $r_x \times r_y = \langle -2y, -2x, 1\rangle$. This does not match our orientation so instead we will use $\langle 2y, 2x, -1\rangle$ and we get that $d\mathbf{S} = \langle 2y, 2x, -1\rangle\ dA$.

We also need to compute $\operatorname{curl}(F) = \nabla \times F = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \times \langle y + \sin(x), z^2 + \cos(y), x^3\rangle = \langle -2z, -3x^2, -1\rangle$. The variables in our parametrization are x and y so we need to rewrite $\operatorname{curl}(F)$ in terms of x and y using the parametrization so we replace z with $2xy$ to get that $\operatorname{curl}(F)(r(x, y)) = \langle -4xy, -3x^2, -1\rangle$.

If D is the disk $x^2 + y^2 \leq 1$ then $\iint_S \operatorname{curl}(F) \cdot\ d\mathbf{S} = \iint_D \langle -4xy, -3x^2, -1\rangle \cdot \langle 2y, 2x, -1\rangle\ dA = \iint_D -8xy^2 - 6x^3 + 1\ dA$. This can be integrated by switching to polar, or we can note that $\iint_D -8xy^2\ dA = 0$ and $\iint_D -6x^3\ dA = 0$ by symmetry and $\iint_D 1\ dA = \pi$ as it is the area of a circle of radius 1. This gives us that $\int_C (y + \sin(x))dx + (z^2 + \cos(y))dy + x^3 dz = \iint_S \operatorname{curl}(F) \cdot\ d\mathbf{S} = \pi$.

19. (a) Find a number c such that the force field $F = \langle ye^x + 3x^2 + 3y^2, e^x + cxy + 3y^2\rangle$ is conservative.

    If $P = ye^x + 3x^2 + 3y^2, Q = e^x + cxy + 3y^2$ then F will be conservative if and only if $P_y = Q_x$. The derivatives are $P_y = e^x + 6y$ and $Q_x = e^x + cy$ so $c = 6$.

    (b) Suppose the constant c has the value found in part a. Find a function $f(x, y)$ such that $F = \nabla f$.

    We are trying to find f with $f_x = ye^x + 3x^2 + 3y^2$ and $f_y = e^x + 6xy + 3y^2$. Integrating f_x with respect to x we get that $f(x, y) = ye^x + x^3 + 3xy^2 + g(y)$. The y partial derivative of this is $f_y(x, y) = e^x + 6xy + g'(y)$. We set this equal to our formula for $f_y = e^x + 6xy + 3y^2$ to get that $g'(y) = 3y^2$ so $g(y) = y^3$. Plug this into the formula for $f(x, y)$ to get that $f(x, y) = ye^x + x^3 + 3xy^2 + y^3$.

    (c) Continuing to assume that c has the value found in part a, find the work done by F on a particle moving from $(1, 0)$ to $(0, 1)$ along the circle of radius 1 centered at the origin.

Using the fundamental theorem of line integrals, this will be
$f(0,1) - f(1,0) = 2 - 1 = 1$.

20. A solid sphere with radius $\sqrt{2}$ is cut into two unequal piece by a plane, where the distance from the center of the ball to the plane is 1 unit. Set up, but do not evaluate, integrals equal to the volume of the smaller piece. Do this in rectangular, spherical, and cylindrical coordinates.

    We first need to put the sphere and plane into 3-dimensional space. The easiest way to do this is to put the center of the sphere at the origin so the sphere is $x^2 + y^2 + z^2 = 2$ and take the plane to be $z = 1$. The intersection of these surfaces is the circle $x^2 + y^2 = 1$ on the plane $z = 1$. This means that for rectangular and cylindrical coordinates, the projection to the xy-plane will be the disk $x^2 + y^2 \leq 1$. Note also that if we draw the cross section on the yz-plane it the circle $y^2 + z^2 = 2$ and the line $z = 1$ which intersect at the point $(1,1), (-1,1)$. At the point $(1,1)$ the angle with the positive z-axis at the intersection is $\pi/4$ so when we set up the spherical coordinates the ϕ values will go from 0 to $\pi/4$. The three integrals are:

    Rectangular: $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1}^{\sqrt{2-x^2-y^2}} \quad dzdydx$

    Cylindrical: $\displaystyle\int_{0}^{2\pi} \int_{0}^{1} \int_{1}^{\sqrt{1-r^2}} r\ dzdrd\theta$

    Spherical: $\displaystyle\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{1/\cos(\phi)}^{\sqrt{2}} \rho^2 \sin(\phi)\ d\rho d\phi d\theta$

21. Let S be the surface consisting of three surfaces S_1, S_2, S_3 where S_1 is the part of the cylinder $x^2 + y^2 = 16$ with $0 \leq z \leq 4$, S_2 is the disk $x^2 + y^2 \leq 16$ on the plane $z = 4$, and S_3 is the hemisphere $z = \sqrt{16 - x^2 - y^2}$. Find $\iint_S F \cdot\ d\mathbf{S}$ where $F = \langle e^{\cos(z)}, 2y + 3x, 1/(x^2 + y^2) \rangle$.

    If computed directly, this would be 3 surface integrals and the $e^{\cos(z)}$ component of F would be messy and difficult to integrate. This is a closed surface so we can use the divergence theorem to replace these three surface integrals with one triple integral. No orientation is given for S so we can assume the positive (outward) orientation and by the divergence theorem $\iint_S F \cdot\ d\mathbf{S} = \iiint_E \text{div}F\ dV$ where E is the region enclosed by S.

    The divergence of F is
    $\text{div}F = \nabla \cdot F = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle e^{\cos(z)}, 2y + 3x, 1/(x^2 + y^2) \rangle = 0 + 2 + 0 = 2$.
    The region E is easiest to set up in cylindrical coordinates so we get that

    $$\iiint_E \text{div}F\ dV = \int_{0}^{2\pi} \int_{0}^{4} \int_{\sqrt{16-r^2}}^{4} 2r\ dzdrd\theta$$

$$= \int_0^{2\pi} \int_0^4 8r - 2r\sqrt{16-r^2} \ drd\theta \ .$$

The integral of $8r$ is $4r^2$ and we can use u-substitution with $u = 16 - r^2$ to integrate $-2r\sqrt{16-r^2}$ and get that this equals

$$\int_0^{2\pi} 4r^2 + \frac{2}{3}(16-r^2)^{3/2}\Big|_0^4 \ d\theta = \int_0^{2\pi} \frac{64}{3} \ d\theta = \frac{128\pi}{3} \ .$$

22. Let S be the surface of the region which is between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and also above the cone $z = \sqrt{x^2 + y^2}$. Find $\iint_S F \cdot \ d\mathbf{S}$ where $F = \langle 2x^2 z, xyz, y^4 \rangle$.

    As in the previous problem we will use the divergence theorem to replace 3 surface integrals with 1 triple integral. The divergence of F is $\text{div}F = 4xz + xz + 0 = 5xz$. The region is easiest to set up in spherical coordinates. In spherical coordinates $x = \rho\sin(\phi)\cos(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2 \sin(\phi) \ d\rho d\phi d\theta$ and the spheres have formulas $\rho = 1$ and $\rho = 2$ and the cone has formula $\phi = \pi/4$. So we get the integral

$$\iiint_E \text{div}F \ dV = \int_0^{2\pi} \int_0^{\pi/4} \int_1^2 5\rho^4 \sin^2(\phi)\cos(\phi)\cos(\theta) \ d\rho d\phi d\theta \ .$$

    As the bounds are all constants we can rewrite this as

$$\left(\int_1^2 5\rho^4 \ d\rho\right)\left(\int_0^{\pi/4} \sin^2(\phi)\cos(\phi) \ d\phi\right)\left(\int_0^{2\pi} \cos(\theta) \ d\theta\right) \ .$$

    The last integral is 0 so the whole thing is 0.

23. Evaluate $\int_C x^2 y dx + \frac{1}{3}x^3 dy + xydz$ where C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise when viewed from above.

    We will use Stokes' Theorem with the surface S equal to the part of $z = y^2 - x^2$ which is inside the cylinder $x^2 + y^2 = 1$ oriented upwards. The surface S can be parametrized as $r(x, y) = \langle x, y, y^2 - x^2 \rangle$ for $x^2 + y^2 \leq 1$. The partial derivatives are $r_x = \langle 1, 0, -2x \rangle$ and $r_y = \langle 0, 1, 2y \rangle$ so the cross product $r_x \times r_y = \langle 2x, -2y, 1 \rangle$. This matches our orientation.

    Take $F = \langle x^2 y, \frac{1}{3}x^3, xy \rangle$ so $\text{curl}F = \langle x, -y, 0 \rangle$. This is already just in terms of x, y which are the variables in our parametrization of S so we don't need to do anything else with $\text{curl}F$. So we get that this integral equals

$$\iint_{x^2+y^2\leq 1} \langle x, -y, 0 \rangle \cdot \langle 2x, -2y, 1 \rangle \ dA = \iint_{x^2+y^2\leq 1} 2x^2 + 2y^2 \ dA \ .$$

Changing to polar this becomes

$$\int_0^{2\pi}\int_0^1 2r^3\ drd\theta = \int_0^{2\pi} \frac{1}{2}r^4\Big|_0^1\ d\theta = \int_0^{2\pi}\frac{1}{2}\ d\theta = \pi\ .$$

24. Let S be the top and 4 sides (but not bottom) of the cube with vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)$ oriented outwards. Let $F = \langle xy, z^2y, x^3z\rangle$. Compute $\iint_S F\cdot\ d\mathbf{S}$ and $\iint_S \text{curl}(F)\cdot\ d\mathbf{S}$.

    We will first find $\iint_S F\cdot\ d\mathbf{S}$. This could be done directly but would involve 5 surface integrals so we want to instead use the divergence theorem. The surface is not closed so we will need to close it with the bottom face and compute both the integral over the whole cube and the integral over the bottom face. Let S_1 be the surface of the cube oriented outwards and S_2 be the bottom face oriented down. Then $S_1 = S\cup S_2$ so $\iint_{S_1} F\cdot\ d\mathbf{S} = \iint_S F\cdot\ d\mathbf{S} + \iint_{S_2} F\cdot\ d\mathbf{S}$ so $\iint_S F\cdot\ d\mathbf{S} = \iint_{S_1} F\cdot\ d\mathbf{S} - \iint_{S_2} F\cdot\ d\mathbf{S}$. The divergence of F is $\text{div}F = y + z^2 + x^3$ so by the divergence theorem $\iint_{S_1} F\cdot\ d\mathbf{S} = \int_0^1\int_0^1\int_0^1 y+z^2+x^3\ dzdydx = \frac{13}{12}$. We will compute $\iint_{S_2} F\cdot\ d\mathbf{S}$ directly. The surface S_2 can be parametrized as $r(x,y) = \langle x,y,0\rangle$ for $0\le x\le 1, 0\le y\le 1$. The derivatives are $r_x = \langle 1,0,0\rangle, r_y = \langle 0,1,0\rangle$ so the cross product is $r_x\times r_y = \langle 0,0,1\rangle$. This does not match our orientation so we use $\langle 0,0,-1\rangle$ instead. We also need plug our parametrization into F so we replace z with 0 to get that $F = \langle xy, z^2y, x^3z\rangle = \langle xy,0,0\rangle$. Then $\iint_{S_2} F\cdot\ d\mathbf{S} = \int_0^1\int_0^1\langle xy,0,0\rangle\cdot\langle 0,0,-1\rangle\ dydx = \int_0^1\int_0^1 0\ dydx = 0$. It follows that $\iint_S F\cdot\ d\mathbf{S} = \iint_{S_1} F\cdot\ d\mathbf{S} - \iint_{S_2} F\cdot\ d\mathbf{S} = \frac{13}{12} - 0 = \frac{13}{12}$.

    Next compute $\iint_S \text{curl}(F)\cdot\ d\mathbf{S}$. Again if done directly we would need to do 5 surface integrals. As we are integrating the curl of F, we can use Stokes' Theorem to replace this with the line integral over the boundary curve C. C is the square with vertices $(0,0,0),(1,0,0),(0,1,0),(1,1,0)$ oriented counterclockwise when viewed from above. By Stokes' Theorem, $\iint_S \text{curl}(F)\cdot\ d\mathbf{S} = \int_C F\cdot\ dr = \int_C (xy)dx + (z^2y)dy + (x^3z)dz$. As C is on the xy-plane we will have that $z=0$ so this becomes $\int_C xy\ dx$. If we look at the 4 line segments which make up the square, we have that $dx = 0$ on the two vertical lines and $y=0$ on the bottom line so the integral will be 0 over these three lines. This just leaves the top C_1 from $(1,1)$ to $(0,1)$. We can parametrize C_1 as $x = 1-t, y = 1, 0\le t\le 1$ so $dx = -dt$ and $\int_{C_1}(xy)\ dx = \int_0^1 -(1-t)\ dt = -\frac{1}{2}$. So we get that $\iint_S \text{curl}(F)\cdot\ d\mathbf{S} = -\frac{1}{2}$.

    Another way to compute $\iint_S \text{curl}(F)\cdot\ d\mathbf{S}$ is to replace it with $\iint_{S_1}\text{curl}(F)\cdot\ d\mathbf{S}$ where S_1 is another surface with the same boundary curve. In particular we can take S_1 to be the bottom face oriented up. S_1 can be

parametrized as $r(x, y) = \langle x, y, 0\rangle$ for $0 \leq x \leq 1, 0 \leq y \leq 1$ and $r_x \times r_y = \langle 0, 0, 1\rangle$. Then $\text{curl}F = \langle -2yz, -3x^2z, -x\rangle$ and plugging in our parametrization this becomes $\langle 0, 0, -x\rangle$. So
$\iint_S \text{curl}(F) \cdot \, d\mathbf{S} = \int_0^1 \int_0^1 \langle 0, 0, -x\rangle \cdot \langle 0, 0, 1\rangle \, dA = \int_0^1 \int_0^1 -x \; dxdy = -\frac{1}{2}$.